Since $h$ is twice the area described in the unit of time. Therefore if $T$ be the time of describing the ellipse, we have

$$T = \frac{\text{area of the ellipse}}{\frac{1}{2}h}$$

$$= \frac{\pi ab}{\frac{1}{2}\sqrt{\mu \frac{b^2}{a}}}$$

$$= \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} \quad \text{-----------} (5)$$

So that the square of the periodic time varies as the cube of the major axis.

COR. I. If a particle be projected at a distance $R$ with velocity $V$ in any direction the path is an ellipse, parabolla or hyperbola, according as

$$V^2 <=> \frac{2\mu}{R}$$

Now the square of the velocity that would be acquired in falling from infinity to the distance $R$,

$$= 2\int_{\infty}^{R} \left(-\frac{\mu}{r^2}\right) dr$$

$$= \left[\frac{2\mu}{r}\right]_{\infty}^{R}$$

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Hence the path is an ellipse, parabolla or hyperbola, according as the velocity at any point is $<=>$ that acquired in falling from infinity to the point.

COR. The velocity $V_1$ for the description of a circle of radius $R$ is given by

$$\frac{V_1^2}{R} = \text{normal acceleration} = \frac{\mu}{R^2}$$

So that $V_1^2 = \frac{\mu}{R}$

and $V_1 = \frac{\text{Velocity from infinity}}{\sqrt{2}}$

Now in the previour the branch of the hyperbola described is the one nearest the centre of force.

If the central acceleration be from the centre and if it vary as the inverse square of the distance, the further branch is described. For in this case the equation of motion is

$$\frac{h^2}{p^2}\frac{dp}{dr} = -\frac{\mu}{r^2}$$

$$\therefore \frac{h^2}{p^2}\frac{dp}{dr} = -\frac{2\mu}{r} + c \quad \text{.............(6)}$$

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Now the $(p, r)$ equation of the branch of a hyperbola is

$$\frac{b^2}{p^2} = 1 - \frac{2a}{r}$$

and this always agrees with (1) provided that

$$\frac{h^2}{b^2} = 1 - \frac{2a}{r} = c$$

and this always so that

$$h = \sqrt{\mu \times \text{Semi-Latus-rectum}}$$

and $v^2 = \frac{h^2}{p^2} = \mu\left(\frac{1}{a} - \frac{2}{r}\right)$

Case — Construction of the orbit given the point of projection and the direction and magnitude of the velocity of projection.

Let $S$ be the centre of attraction, $P$ the point of projection $TPT'$ the direction of projection, $V$ the velocity of projection.

(i) Let $V^2 < \frac{2\mu}{SP}$ then the path is an ellipse whose major axis $2a$ is given by the equation

$$V^2 = \mu\left(\frac{2}{R} - \frac{1}{a}\right)$$

where $R = SP$, so that $2a = \frac{2R\mu}{2\mu - V^2R}$

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Draw $PS'$, so that $PS'$ and $PS$ are on the same side of $TPT'$ making $\angle TPS' = \angle TPS$ and take $PS' = 2a - SP = 2a - R = \frac{V^2R^2}{2\mu - V^2R}$

Then $S'$ is the second focus and the ellipse path is therefore known.

(ii) Let $V^2 = \frac{2\mu}{SP'}$ so that the path is a parabola. We draw the direction $PS'$ as in (i) we get the direction of the axis of the parabola. We draw $SU$ parallel to $PS'$ to meet $TPT'$ in $U$. Now we draw $SY$ ~~parallel~~ perpendicular to $TPT'$ and $YA$ perpendicular to $SU$. Then $A$ is the vertex of the required parabolla whose focus is $S$ and the curve can be constructed.

The semi-latus-rectum $= 2SA = 2\frac{SY^2}{SP} = \frac{2p_0^2}{R}$

where $p_0$ is the perpendicular from $S$ on the direction of projection.

(iii) Let $V^2 > \frac{2\mu}{SP'}$, so that the path is a hyperbola of transverse axis $2a$ is given by the ~~ellipse~~ equation

$$V^2 = \mu\left(\frac{2}{R} + \frac{1}{a}\right), \text{ and hence } 2a = \frac{2\mu R}{V^2R - 2\mu}$$

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